Distinguishing non-archimedean seminorms

by

W.H. Schikhof

§1. Introduction and summary. (For terminology, see §2) In [2] the following separation theorem is proved. *Let* K *be spherically complete and let* E *be a locally convex space over* K. *If* $A \subset E$ *is closed and absolutely convex and if* $x \in E \backslash A$ *then there is an* $f \in E'$ *such that*

$$(*) \qquad |f(A)| \le 1 \quad , \quad f(x) = 1.$$

In order to obtain, 'real' separation one would prefer to have

$$(**) \qquad |f(A)| < 1 \quad , \quad f(x) = 1$$

rather than (*). However, with the techniques used in [2], it is not clear how to arrive at such a result for densely valued fields K. The main obstruction is the fact that for an open absolutely convex, absorbing set A its associated seminorm q_A,

$$q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \quad (x \in E)$$

does not determine A; one only has the rather 'vague' relation

$$\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \le 1\}.$$

(Example. The open and closed convex sets $\{(\xi_1,\xi_2) \in K^2 : |\xi_1| \le 1, |\xi_2| \le 1\}$, $\{(\xi_1,\xi_2) \in K^2 : |\xi_1| \le 1, |\xi_2| < 1\}$, $\{(\xi_1,\xi_2) \in K^2 : |\xi_1| < 1, |\xi_2| < 1\}$ all have the associated (semi)norm $(\xi_1,\xi_2) \mapsto \max(|\xi_1|, |\xi_2|)$.)

In this paper we shall extend the notion of a seminorm by admitting a larger range yielding the notion of a distinguishing seminorm (Definition 2). We shall prove that each absolutely convex absorbing set can be written as $\{x \in E : p(x) < 1\}$ for some distinguishing seminorm p (Theorem 5). Next, we shall prove a Hahn Banach theorem for linear functions majorized by distinguishing seminorms (Theorem 6) and shall obtain, as a corollary, a strong separation theorem (with (**) in place of (*)). Here the notion of distinguishing seminorm is used only in the proof, not in the formulation (Theorem 8).

Note. This separation theorem can (for Banach spaces) also be obtained as a corollary of [1], Theorem 6.21. However, one techniques differ very much from the ones used in the (clever) proof of [1]. Also, the notion of a distinguishing seminorm may very well yield new applications.

§2. Terminology

Throughout K is a non-archimedean complete non-trivially valued field with valuation $|\ |$, $|K| := \{|\lambda| : \lambda \in K\}$. E is a vector space over K. A seminorm on E is a map $q : E \to [0,\infty)$ satisfying (i) $q(0) = 0$, (ii) $q(\lambda x) = |\lambda| q(x)$ $(x \in E, \lambda \in K)$, (iii) $q(x+y) \le q(x) \vee q(y)$ $(x,y \in E)$ where $\vee$ indicates 'maximum'.

A subset A of E is absolutely convex if it is a module over the ring $\{\lambda \in K : |\lambda| \le 1\}$, convex if it is an additive coset of an absolutely convex set, absorbing if $\bigcup_{\lambda \in K} \lambda A = E$.

Each convex subset of K has the form $\{\lambda \in K : |\lambda-\alpha| < r\}$ or $\{\lambda \in K : |\lambda-\alpha| \le r\}$ for some $\alpha \in K$, $r \in [0,\infty]$.

For a seminorm q the sets $\{x \in E : q(x) < 1\}$ and $\{x \in E : q(x) \le 1\}$ are absolutely convex and absorbing. Conversely, for an absorbing, absolutely convex subset A of E the <u>associated seminorm</u> q_A defined by

$$q_A(x) := \inf\{|\lambda| : \lambda \in K : x \in \lambda A\} \qquad (x \in E)$$

is a seminorm satisfying

$$\{x \in E : q_A(x) < 1\} \subset A \subset \{x \in E : q_A(x) \le 1\}.$$

K is <u>spherically complete</u> if for any collection $\mathcal{C}$ of convex subsets for which $A,B \in \mathcal{C} \Rightarrow A \cap B \neq \emptyset$ we have $\cap\mathcal{C} = \emptyset$. A <u>locally convex space</u> is a K-vector space E with a topology induced by a collection of seminorms. Its dual space is denoted E'.

§3. <u>Distinguishing seminorms</u>

We enlarge the set $[0,\infty)$ by giving each positive real number a an immediate predecessor a^-. Formally

$$V := [0,\infty) \cup (0,\infty)^-$$

where $(0,\infty)^- := \{a^- : a \in (0,\infty)\}$ is a second copy of $(0,\infty)$. Further we define $0^- := 0$. The formulas

$$a^- < b \Leftrightarrow a \le b \qquad (a,b \in (0,\infty))$$

$$a^- < b^- \Leftrightarrow a < b^- \Leftrightarrow a < b \qquad (a,b \in (0,\infty))$$

$$0 < a \qquad (a \in V,\ a \neq 0)$$

define a linear ordering < on V extending the usual ordering on $[0,\infty)$.

The projection map $\pi : V \to [0,\infty)$ is defined by

$$\pi(a) := a \qquad (a \in [0,\infty))$$

$$\pi(a^{\sim}) := a \qquad (a \in [0,\infty)).$$

Finally we extend the multiplication on $[0,\infty)$ to a multiplication $[0,\infty) \times V \to V$ by requiring $a \cdot b^{\sim} := (ab)^{\sim}$ for all $a,b \in [0,\infty)$.

Proposition 1 collects some direct consequences from the definitions.

PROPOSITION 1.

(i) Let $b,c \in V$, $b \le c$. Then $ab \le ac$ for all $a \in [0,\infty)$.

(ii) $\pi(a \vee b) = \pi(a) \vee \pi(b) \qquad (a,b \in V)$.

(iii) $\pi(ab) = a\pi(b) \qquad (a \in [0,\infty),\ b \in V)$.

DEFINITION 2. A distinguishing seminorm on E is a map $p : E \to V$ satisfying

(i) $p(0) = 0$

(ii) $p(\lambda x) = |\lambda|\, p(x) \qquad (\lambda \in K,\ x \in E)$

(iii) $p(x+y) \le p(x) \vee p(y) \qquad (x,y \in E)$.

If, in addition, $p(x) = 0$ implies $x = 0$ then p is a distinguishing norm.

Examples

1. Any seminorm on E.

2. The map $(\xi_1,\xi_2) \mapsto |\xi_1| \vee |\xi_2|^{\sim}$ $((\xi_1,\xi_2) \in K^2)$ is a distinguishing norm on K^2.

3. Let E be the space of all bounded K-valued functions on a set X. Then

$$\|f\|_{\infty}^{\sim} := \sup\{|f(x)| : x \in X\} \qquad (f \in E),$$

where the supremum is taken in V, defines a distinguishing norm on E. Observe that for $f \in E$ we have $\|f\|_{\infty}^{\sim} \le 1$ if and only if $|f(x)| \le 1$ for all $x \in E$ but

$$\|f\|_\infty^\sim < 1 \Leftrightarrow \|f\|_\infty^\sim \le 1^- \Leftrightarrow |f(x)| < 1 \text{ for all } x \in E$$

so that $\|f\|_\infty^\sim = 1$ if and only if $1 = \max\{|f(x)| : x \in X\}$.

PROPOSITION 3. *Let p be a distinguishing seminorm on E. Then $\pi \circ p$ is a seminorm on E.*

Proof. The statement follows directly from Proposition 1 (ii), (iii).

DEFINITION 4. Let $A \subset E$ be absorbing and absolutely convex, and let q_A be its associated seminorm. The *distinguishing seminorm* p_A, *associated to* A is

$$p_A(x) := \begin{cases} q_A(x)^- & \text{if } q_A(x) = \min\{|\lambda| : \lambda \in K,\ x \in \lambda A\} \\ q_A(x) & \text{otherwise.} \end{cases}$$

The following theorem shows that the associated distinguishing seminorm of an absorbing absolutely convex set A determines A.

THEOREM 5. *Let $A \subset E$ be absorbing and absolutely convex. Let p_A and q_A be as in Definition 4. Then p_A is a distinguishing seminorm with $\pi \circ p_A = q_A$. Further we have*

$$A = \{x \in E : p_A(x) < 1\} = \{x \in E : p_A(x) \le 1^-\}.$$

Proof. Clearly $\pi \circ p_A = q_A$. We first check the equality $A = \{x \in E : p_A(x) < 1\}$. Let $p_A(x) < 1$. If $p_A(x) < 1^-$ then $q_A(x) < 1$, so $x \in A$. If $p_A(x) = 1^-$ then $1 \in \{|\lambda| : \lambda \in K : x \in \lambda A\}$ so that $x \in \mu A$ for some $\mu \in K$, $|\mu| = 1$. By absolute convexity, $x \in \mu^{-1}\mu A = A$. Conversely, let $x \in A$. Then $q_A(x) \le 1$ so that $p_A(x) \le q_A(x) \le 1$. If $p_A(x) = 1$ then $1 = \inf\{|\lambda| : x \in \lambda A\}$ is not a minimum contradicting $x \in A$. Hence,

$p_A(x) < 1$. Finally we prove the conditions (i), (ii), (iii) of Definition 2 for a distinguishing seminorm for $p = p_A$. We have $p_A(0) = 0^- = 0$. To prove (ii) we may assume $\lambda \neq 0$ and $p_A(x) \neq 0$. If $p_A(x) \in (0,\infty)$ then $q_A(x) = \inf\{|\tau| : x \in \tau A\}$ is not a minimum. Then neither is $q_A(\lambda x) = \inf\{|\tau| : \lambda x \in \tau A\}$ so that $p_A(\lambda x) = q_A(\lambda x) = |\lambda| q_A(x) = |\lambda| p_A(x)$. If $p_A(x) \in (0,\infty)^-$ then $q_A(x) = \min\{|\tau| : x \in \tau A\}$. Then also $q_A(\lambda x) = \min\{|\tau| : \lambda x \in \tau A\}$ so that $p_A(\lambda x) = q_A(\lambda x)^- = (|\lambda| q_A(x))^- = |\lambda| q_A(x)^- = |\lambda| p_A(x)$. For the proof of the strong triangle inequality (iii) we may assume $p_A(x+y) \neq 0$. We distinguish two cases.

(i) $p_A(x+y) \in (0,\infty)^-$. By Proposition 1 (ii) and $\pi \circ p_A = q_A$, from

$$q_A(x+y) \leq q_A(x) \vee q_A(y)$$

we obtain $p_A(x+y) = q_A(x+y)^- \leq q_A(x)^- \vee q_A(y)^- \leq p_A(x) \vee p_A(y)$.

(ii) $p_A(x+y) \in (0,\infty)$. Then the valuation of K is dense. Assume $p_A(y) \leq p_A(x)$. Suppose $p_A(x+y) > p_A(x)$; we derive a contradiction. There is a $\lambda \in K$ such that $p_A(x+y) \geq |\lambda| > p_A(x)$. (In fact, if $p_A(x) \neq p_A(x+y)^-$ then the interval $(p_A(x), p_A(x+y))$ contains infinitely many elements of $|K|$, if $p_A(x) = p_A(x+y)^-$ then $p_A(x+y) \in |K|$ and we may choose $\lambda \in K$ such that $|\lambda| = p_A(x+y)$.) Then $p_A(\lambda^{-1}y) \leq p_A(\lambda^{-1}x) < 1$ so that $\lambda^{-1}y \in A$, $\lambda^{-1}x \in A$ and, by absolute convexity, also $\lambda^{-1}(x+y) \in A$ implying $p_A(x+y) < |\lambda|$, a contradiction.

§4. Hahn-Banach Theorem

For $\alpha \in K$, $a \in V$ we write $B(\alpha,a) = \{\lambda \in K : |\lambda-\alpha| \leq a\}$.

THEOREM 6. (Hahn-Banach Theorem) *Let K be spherically complete, let p* *be a distinguishing seminorm on a K-vector space E. Let D be a K-linear subspace of E, let f : D → K be a K-linear map satisfying* $|f(d)| \leq p(d)$ $(d \in D)$. *Then f can be extended to a K-linear* $\overline{f} : E \to K$ *such that* $|\overline{f}(x)| \leq p(x)$ $(x \in D)$.

Proof. (It consists of checking that replacing of a seminorm by a distinguishing seminorm does not harm the well-known proof.) A simple application of Zorn's Lemma reduces the problem to the case $E = \{\lambda x + d : \lambda \in K, d \in D\}$ for some $x \in E \setminus D$. We are done if we can choose $\overline{f}(x) = \xi \in K$ such that

$$|\lambda\xi + f(d)| \leq p(\lambda x + d) \qquad (\lambda \in K,\ d \in D).$$

As this condition is satisfied for $\lambda = 0$ and all $d \in D$ and since for $\lambda \neq 0$

$$p(\lambda x+d) = |\lambda|\ p(x+\lambda^{-1}d)$$
$$|\lambda\xi+f(d)| = |\lambda|\ |\xi+f(\lambda^{-1}d)|$$

it suffices, by Proposition 1 (i) to produce a $\xi \in K$ such that

$$|\xi - f(d)| \leq p(x-d) \qquad (d \in D),$$

i.e. we must have that

$$\bigcap_{d \in D} B(f(d),\ p(x-d)) \neq \emptyset.$$

By spherical completeness it suffices to show that for any $d_1, d_2 \in D$ we have $B(f(d_1),\ p(x-d_1)) \cap B(f(d_2),\ p(x-d_2)) \neq \emptyset$. But this follows easily from $|f(d_1) - f(d_2)| \leq p(d_1-d_2) \leq p(d_1-x) \vee p(x-d_2)$.

A typical application: *let* $\ell^\infty := \{(\xi_1,\xi_2,\ldots) : \xi_n \in K$ *for all* n, $\sup|\xi_n| < \infty\}$.

Let K be spherically complete. Then there is a linear function $g : \ell^\infty \to K$

<u>of norm</u> 1 <u>such that</u> (i) $g((\xi_1,\xi_2,\ldots)) = \lim_{n\to\infty} \xi_n$ <u>if</u> $\lim_{n\to\infty} \xi_n$ <u>exists and</u> (ii) $|g((\xi_1,\xi_2,\ldots))| < 1$ <u>if</u> $|\xi_n| < 1$ <u>for all</u> $n \in \mathbb{N}$.

(Proof: Choose in the above theorem $D := c$ (the space of the convergent sequences), $f((\xi_1,\xi_2,\ldots)) := \lim_{n\to\infty} \xi_n$ $((\xi_1,\xi_2,\ldots) \in c)$, and $p((\xi_1,\xi_2,\ldots)) := \sup|\xi_n|$, where the sup is taken in V. Take $g := \overline{f}$.)

§5. <u>Separation of convex sets</u>

Throughout §5, let E be a locally convex space over K. We shall need the following observation.

<u>PROPOSITION 7</u>. <u>An open convex subset of</u> E <u>is closed</u>.

<u>Proof</u>. Any convex set is a coset of an absolutely convex set. An open absolutely convex set is the complement of a union of cosets.

<u>Theorem 8</u>. <u>Let</u> K <u>be spherically complete</u>. <u>Let</u> $A \subset E$ <u>be closed, absolutely convex and let</u> $x \in E\backslash A$. <u>Then there exists an</u> $f \in E'$ <u>such that</u> $|f(A)| < 1$ <u>and</u> $f(x) = 1$.

<u>Proof</u>. There is an absolutely convex open neighbourhood U of 0 such that $(x+U) \cap A = \emptyset$. Then U+A is absolutely convex, open, hence closed (Proposition 7). Further, $x \notin U+A$. Thus, we may assume that A is open and closed. Then A is absorbing. Let p_A be the distinguishing seminorm of A, let $D := \{\lambda x : \lambda \in K\}$ and define $g : D \to K$ by $g(\lambda x) := \lambda$ $(\lambda \in K)$. Then $g(x) = 1$. Since

$$A = \{y \in E : p_A(y) < 1\}$$

(Theorem 5) and $x \notin A$ we have $p_A(x) \geq 1$ so that for $\lambda \in K$ $p_A(\lambda x) = |\lambda|\, p_A(x) \geq |\lambda| = |g(\lambda x)|$, i.e. $|g| \leq p_A$ on D. By Theorem 6 g extends to a linear $f : E \to K$ such that $|f(y)| \leq p_A(y)$ for all $y \in E$.

We have $f(x) = g(x) = 1$ and, for $y \in A$, $|f(y)| \leq p_A(y) < 1$. The continuity of f follows from the continuity of $\pi \circ p_A$ and the inequality $|f| \leq \pi \circ p_A$.

COROLLARY 9. *Let K be spherically complete. Each closed convex set is weakly closed.*

Let A, B be convex subsets of E. If $f : E \to K$ is a linear function then $f(A)$ and $f(B)$ are convex in K. Hence, if $f(A) \cap f(B) = \emptyset$ then $\mathrm{dist}(f(A), f(B)) > 0$. With this in mind the following definition is quite natural.

DEFINITION 10. Two convex subsets A, B of E are *separated* by an $f \in E'$ if $f(A) \cap f(B) = \emptyset$.

If A and B are separated by $f \in E'$ then, since $\mathrm{dist}(f(A), f(B)) > 0$ there is an open convex neighbourhood U of 0 such that $(A+U) \cap B = \emptyset$ (if E is a normed space this is equivalent to $\mathrm{dist}(A,B) > 0$). To prove the converse we need spherical completeness.

THEOREM 11. *Let K be spherically complete. Let A, B be convex subsets of E and suppose there is an open convex neighbourhood U of A such that $(A+U) \cap B = \emptyset$ (observe that this condition is satisfied if A is open). Then A and B can be separated by some $f \in E'$.*

Proof. We may assume that A is open. Let $C := A-B$. Then $0 \notin C$ and $C = \bigcup_{b \in B} (A-b)$ is open, convex. Choose $c \in -C$. Then $T := c + C$ is absolutely convex, open, hence closed and $c \notin T$. By Theorem 8 there is an $f \in E'$ such that $f(c) = 1$, $|f(T)| < 1$. Thus, for each $a \in A$, $b \in B$ we have

$$1 > |f(c + a - b)| = |1 + f(a) - f(b)|.$$

It follows that $|f(a) - f(b)| = 1$ for all $a \in A$, $b \in B$. In particular, $f(A) \cap f(B) = \emptyset$.

References.

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